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Quadratic Spline Interpolation and the Sharpness of Lebesgue's Inequality*

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1. INTRODUCTION

If (P_n) is an unbounded sequence of bounded linear projectors on a normed linear space X, then Lebesgue's inequality

$$\operatorname{dist}(f, \operatorname{ran} P_n) \leqslant \| f - P_n f \| \leqslant \| 1 - P_n \| \operatorname{dist}(f, \operatorname{ran} P_n) \tag{1}$$

leaves open the question as to whether, for a particular f, the interpolation error is of the same order as the best possible error. Specifically, while (1) implies, for a complete X, the existence of some $f \in X$ for which $P_n f$ fails even to converge to f, it may happen that nevertheless for "smoother" f,

$$\limsup_{n \to \infty} \| f - P_n f \| / \operatorname{dist}(f, \operatorname{ran} P_n) < \infty.$$

We consider this question here in the case when $X = C[0, 1]$ and smoothness of $f \in X$ is measured by the number of its derivatives. In this context, it is possible to identify one particular "cause" for the unboundedness of (P_n) which affects also the convergence rate of $\| f - P_n f \|$ for smooth f.

The examples are taken in part from a report by Daniel [3] concerning specific projectors onto quadratic splines. In fact, this note is a reaction to Daniel's report, specifically to his assertion that quadratic spline interpolation at knots gives $O(h^3)$ accuracy for sufficiently smooth f and to his banishment of what he calls "the extremely tedious details of our computations of the errors" to an appendix.

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2. Quadratic Spline Interpolation

For given $\mathbf{t} := (t_i)_1^{n+k}$, nondecreasing with $t_i < t_{i+k}$, all i, let $\mathscr{S}_{k,\mathbf{t}}$ denote the collection of all *splines of order k with knot sequence* $\mathbf{t}$, i.e., the collection of all functions of the form

$$\sum_{i=1}^{n} \alpha_i N_{i,k}$$

for some $\alpha \in \mathbb{R}^n$ and with the kth order B-spline $N_{i,k}$ given by

$$N_{i,k}(t) := ([t_{i+1},..., t_{i+k}] - [t_i,..., t_{i+k-1}])(\cdot - t)_+^k .$$

Here, $[\rho_0,..., \rho_r]f$ denotes the rth divided difference of the function f at the points $\rho_0,..., \rho_r$.

It is convenient to restrict attention to the interval $[t_k, t_{n+1}]$ and to consider further only the specific choice

$$[t_k, t_{n+1}] = [0, 1].$$

For functions f defined on [0, 1], we use

$$\| f \|_\infty := \sup_{t\in[0,1]} | f(t) | .$$

We recall for later use the existence of const so that, for $j = 0, 1, 2$ and all $f \in C^{(j)}[0, 1]$,

$$\operatorname{dist}_\infty(f, \mathscr{S}_{k,\mathbf{t}}) := \inf_{g\in\mathscr{S}_{k,\mathbf{t}}} \| f - g \|_\infty \leqslant \text{const}\, h^j \omega(f^{(j)}; h),$$

with $\omega(g; \cdot)$ the modulus of continuity of g and

$$h := \max_i \Delta t_i .$$

Let $\boldsymbol{\tau} := (\tau_i)_1^n$ be a nondecreasing sequence in [0, 1]. For sufficiently smooth f, the sequence

$$f|_{\boldsymbol{\tau}} := (f_i)_1^n$$

with

$$f_i := f^{(j)}(\tau_i) \qquad \text{for} \quad j := j(i) := \max\{r \mid \tau_{i-r} = \tau_i\}$$

is well defined. We will say that *two functions f and g agree at* $\boldsymbol{\tau}$ provided

$$f|_{\boldsymbol{\tau}} = g|_{\boldsymbol{\tau}} .$$

We are concerned with the spline interpolation problem:

Given f, find $Pf \in \mathscr{S}_{k,\mathbf{t}}$ *which agrees with f at* $\boldsymbol{\tau}$

in the special case $k = 3$ of quadratic spline interpolation. According to a slight extension of the Schoenberg–Whitney theorem [6], Pf is well defined for arbitrary f (for which $f|_\tau$ exists) if and only if

$$\tau_i \in (t_i, t_{i+3}), \qquad i = 1,..., n, \tag{2.1}$$

a condition we will assume from now on. Note that we have suppressed the urge to write more explicitly

$$P_{\mathbf{t},3,\boldsymbol{\tau}}$$

instead of P even though P depends on $\mathbf{t}$, $\boldsymbol{\tau}$ and our choice $k = 3$.

Suppose that $\tau_i < \tau_{i+1}$, all i. Then P is a bounded linear projector on $C[0, 1]$. Further, by [2, Lemma in Sect. 2],

$$\| P \| \geqslant \text{const} \max_i d_i/\Delta\tau_i \tag{2.2a}$$

with

$$d_i := \min\{t_{j+2} - t_j \mid (t_j, t_{j+2}) \cap (\tau_i, \tau_{i+1}) \neq \varnothing\} \tag{2.2b}$$

and const independent of $\mathbf{t}$ and $\boldsymbol{\tau}$. It follows that $\| P \|$ can be made arbitrarily large (even for fixed n) by appropriate choice of $\mathbf{t}$ and $\boldsymbol{\tau}$.

To give a specific example, think of $\boldsymbol{\tau}$ as having been given first, with $\tau_1 = 0$, $\tau_n = 1$, and $\tau_i < \tau_{i+1}$, all i, and that $\mathbf{t}$ has been constructed from $\boldsymbol{\tau}$ by

$$\begin{aligned} t_1 &= t_2 = t_3 = 0, \\ t_{i+2} &= (\tau_i + \tau_{i+1})/2, \qquad i = 2,..., n-2, \\ t_{n+1} &= t_{n+2} = t_{n+3} = 1, \end{aligned} \tag{2.3}$$

i.e., the interior knots are chosen halfway between data points (except for the first and the last pair of data points). Then (2.2) implies

$$\| P \| \geqslant \text{const} \max_i (\min\{\tau_{i+2} - \tau_i, \tau_{i+1} - \tau_{i-1}\}/(\tau_{i+1} - \tau_i)), \tag{2.4}$$

hence $\| P \|$ can be made as large as desired (even for fixed n), e.g., by choosing $\boldsymbol{\tau}$ uniformly spaced and then moving just one τ_i very close to its neighbor τ_{i+1}. In this situation, Lebesgue's inequality would give no useful information about $\|f - Pf\|_\infty$ for smooth f. In fact, we can prove that

$$\|f - Pf\|_\infty \leqslant 4.5\, h\omega(f^{(1)}; h) \qquad \text{for all} \quad f \in C^{(1)}[0, 1] \tag{2.5}$$

regardless of whether $\| P \|$ can be bounded.

In order to prove (2.5) and in preparation for further examples, we consider the map P' given on $C[0, 1]$ by the rule

$$P'f^{(1)} = (Pf)^{(1)}, \qquad \text{all} \quad f \in C^{(1)}[0, 1]. \tag{2.6}$$

Let $\mathbf{t}' := (t_i)_2^{n+2}$ and let λ_i be the linear functional given by the rule

$$\lambda_i g := (1/\Delta\tau_{i-1}) \int_{\tau_{i-1}}^{\tau_i} g(s)\, ds.$$

Then P' is a linear projector with range $\mathscr{S}_{2,\mathbf{t}'}$ and interpolation conditions span $(\lambda_i)_2^n$, i.e., for given g, $P'g$ is the unique element in $\mathscr{S}_{2,\mathbf{t}'}$ satisfying

$$\lambda_i P'g = \lambda_i g, \qquad i = 1,..., n.$$

Since $\| \sum_{i=2}^n a_i N_{i,2} \|_\infty = \| a \|_\infty$ while $\| \sum_{i=2}^n b_i \lambda_i \| = \| b \|_1$, it follows from [1, Corollary 2] that

$$\| P' \| = \| A^{-1} \|_\infty \tag{2.7}$$

with the $(n - 1) \times (n - 1)$ Gramian matrix A given by

$$A := (\lambda_i N_{j,2})_{i,j=2}^n . \tag{2.8}$$

Now, for $\mathbf{t}$ and $\boldsymbol{\tau}$ as related by (2.3), A turns out to be tridiagonal, of the form

$$(Ax)_i = \frac{1}{4} \begin{cases} (2 + 2(1 - \alpha_2))\, x_2 + 2\alpha_2 x_3, & i = 2, \\ (1 - \alpha_{i-1})\, x_{i-1} + (2 + \alpha_{i-1} + (1 - \alpha_i))\, x_i + \alpha_i x_{i+1}, & i = 3,..., n - 1, \\ 2(1 - \alpha_{n-1})\, x_{n-1} + (2 + 2\alpha_{n-1})\, x_n, & i = n, \end{cases} \tag{2.9a}$$

with

$$\alpha_i := \begin{cases} 2(\tau_2 - \tau_1)/(\tau_2 + \tau_3 - 2\tau_1), & i = 2, \\ (\tau_i - \tau_{i-1})/(\tau_{i+1} - \tau_{i-1}), & i = 3,..., n - 2, \\ (\tau_{n-1} - \tau_{n-2})/(2\tau_n - \tau_{n-1} - \tau_{n-2}), & i = n - 1. \end{cases} \tag{2.9b}$$

Although A fails to be strictly row diagonally dominant in general, the following variant of the standard argument establishes that

$$\| A^{-1} \|_\infty \leqslant 8. \tag{2.10}$$

For given $x = (x_i)_2^n$, let i be such that $| x_i | = \| x \|_\infty$ and assume without loss that $x_i > 0$. If both x_{i-1} and x_{i+1} are nonnegative then (2.9) implies that

$$\| Ax \|_\infty \geqslant (Ax)_i \geqslant x_i/2 = \| x \|_\infty/2.$$

Otherwise, assume without loss that $x_{i-1} < 0$. If now

$$| x_{i-1} | \leqslant | x_i |/2,$$

then, for $i < n$,

$$\begin{aligned} 4\,|(Ax)_i| &\geqslant (2 + \alpha_{i-1} + (1 - \alpha_i))\, x_i - (1 - \alpha_{i-1})\, | x_{i-1} | - \alpha_i\, | x_{i+1} | \\ &\geqslant (2 + \alpha_{i-1} - (1 - \alpha_{i-1})/2 + 1 - 2\alpha_i)\, x_i \\ &\geqslant x_i/2, \end{aligned}$$

hence then

$$\| Ax \|_\infty \geqslant \| x \|_\infty/8.$$

For $i = n$, the terms look slightly different, but the conclusion is the same. Finally, if $| x_{i-1} | \geqslant | x_i |/2$, then, for $3 < i < n$,

$$\begin{aligned} &4\,|(Ax)_{i-1}| + 4\,|(Ax)_i| \\ &\quad\geqslant (2 + \alpha_{i-2} + (1 - \alpha_{i-1}))\, | x_{i-1} | - (1 - \alpha_{i-2})\, | x_{i-2} | - \alpha_{i-1} x_i \\ &\qquad + (2 + \alpha_{i-1} + (1 - \alpha_i))\, x_i - (1 - \alpha_{i-1})\, | x_{i-1} | - \alpha_i\, | x_{i+1} | \\ &\quad\geqslant (2 + \alpha_{i-2})\, | x_{i-1} | + (2 + (1 - \alpha_i) - (1 - \alpha_{i-2}) - \alpha_i)\, x_i \\ &\quad\geqslant 2\, | x_{i-1} | \geqslant | x_i |, \end{aligned}$$

hence, again $\| Ax \|_\infty \geqslant \| x \|_\infty/8$, while the analogous argument establishes

$$\begin{aligned} \| Ax \|_\infty &\geqslant 3\, \| x \|_\infty/16 \qquad \text{if} \quad i = n, \\ \| Ax \|_\infty &\geqslant \| x \|_\infty/8 \qquad \text{if} \quad i = 3. \end{aligned}$$

This proves, by Lebesgue's inequality and (2.7) and (2.10), that, for the choice (2.3),

$$\| f^{(1)} - P'f^{(1)} \|_\infty \leqslant 9 \operatorname{dist}_\infty(f^{(1)}, \mathscr{S}_{2,\mathbf{t}'})$$

and therefore, integrating once and noting that $f - Pf$ vanishes at the τ_i's, and that $h = \max_i \Delta t_i = \frac{1}{2}\max_i(\tau_i - \tau_{i-2}) \in \{\max_i \Delta\tau_i/2, \max_i \Delta\tau_i\}$, we have proved the following proposition.

PROPOSITION 1. *If* $\mathbf{t}$ *and* $\boldsymbol{\tau}$ *are related by* (2.3) *and* $\boldsymbol{\tau}$ *is strictly increasing, then parabolic spline interpolation satisfies*

$$\| f - Pf \|_\infty \leqslant 9h \operatorname{dist}_\infty(f^{(1)}, \mathscr{S}_{2,\mathbf{t}'})$$

for all $f \in \mathbb{L}_\infty^{(1)}[0, 1]$.

In this example, then, going to a smooth subset (viz., $\mathbb{L}_\infty^{(1)}[0, 1]$ of $C[0, 1]$)

has removed entirely the influence of a "large" $\| P \|$ on the convergence rate. Naively, P takes its norm only on relatively unsmooth functions.

We note that the phrase "and $\boldsymbol{\tau}$ is strictly increasing" in the hypothesis of Proposition 1 can be relaxed to "and $\tau_i < \tau_{i+2}$, all i" provided that f is further constrained to satisfy

$$\tau_{i-1} = \tau_i \qquad \text{implies} \qquad | \lambda_i f^{(1)} | = | f^{(1)}(\tau_i) | \leqslant \| f^{(1)} \|_\infty .$$

Matters are different in the following example, mentioned by Daniel [3], of quadratic spline interpolation at knots, i.e.,

$$\tau_i = t_{i+1}, \qquad i = 1, \ldots, n \tag{2.11}$$

where again

$$t_1 = t_2 = t_3 = 0, \qquad t_{n+1} = t_{n+2} = t_{n+3} = 1.$$

We choose $\mathbf{t}$ (essentially) equispaced, i.e.,

$$\Delta t_i = h, \qquad i = 3, \ldots, n. \tag{2.12}$$

Now the Gramian A for P' (cf. (2.8)) turns out to be lower triangular and bidiagonal,

$$A = \begin{bmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & \frac{1}{2} & \frac{1}{2} & & \\ & & \ddots & \ddots & \\ & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

In order to compute $\| A^{-1} \|_\infty$, we recall from the argument for [1, Lemma 3] that A is totally positive (for any choice of $\mathbf{t}$ and $\boldsymbol{\tau}$). This implies (as in [1, Corollary to Lemma 3]) that

$$\| x \|_\infty / \| Ax \|_\infty \leqslant \| A^{-1} \|_\infty \leqslant \| x \|_\infty / \min_i |(Ax)_i| \tag{2.13}$$

whenever x is a vector for which $\min_i (-)^i (Ax)_i > 0$. Choose, in particular,

$$x_i = (-)^i (2i - 3), \qquad i = 2, \ldots, n.$$

Then $\| x \|_\infty = 2n - 3$ and

$$(Ax)_i = (-)^i, \qquad i = 2, \ldots, n,$$

hence (2.13) implies

$$\| A^{-1} \|_\infty = 2n - 3$$

showing, with (2.7), that

$$\| P' \| = O(h^{-1})$$

in this case. Lebesgue's inequality now gives

$$\| f^{(1)} - P'f^{(1)} \|_\infty \leqslant \text{const}\, h \, \| f^{(3)} \|_\infty \qquad \text{for} \quad f \in C^{(3)}[0, 1], \tag{2.14}$$

which is to be compared with

$$\text{dist}_\infty(f^{(1)}, \mathscr{S}_{2,\mathbf{t}'}) \leqslant \text{const}\, h^2 \, \| f^{(3)} \|_\infty \qquad \text{for} \quad f \in C^{(3)}[0, 1].$$

In short, we have again an apparent loss in the convergence rate for a smooth function. But in this case, the loss is (essentially) irreparable, i.e., (2.14) gives the best possible rate when considering all $f \in C^{(3)}[0, 1]$.

Precisely, there exists a linear projector Q' with the *same* interpolation conditions as those of P' (but with a different range, of course) which is bounded on $C[0, 1]$ independently of h and for which the map

$$h^2 D^2 Q' : g \mapsto h^2 (Q'g)^{(2)} \tag{2.15}$$

is bounded independently of h (as a map from $C[0, 1]$ to $\mathbb{L}_\infty[0, 1]$). For example, one can take Q' as given by

$$Q'f^{(1)} = (Qf)^{(1)} \qquad \text{all } f \in C^{(1)}[0, 1],$$

with Q cubic spline interpolation at the knots $t_2, \ldots, t_{n+1}$ with the subsidiary condition that t_n is not a knot for Qf, i.e., $\text{jump}_{t_n}(Qf)^{(3)} = 0$. Q' is bounded on $C[0, 1]$ independently of h (as can be deduced from Sharma and Meir [7]) and the boundedness of the map $h^2 D^2 Q'$ follows from that by Markov's inequality. This implies that

$$\begin{aligned}(1 - P')g &= (1 - P')\, Q'g + (1 - P')(1 - Q')g \\ &= (1 - P')\, Q'g + (1 - Q')g,\end{aligned}$$

since P' and Q' satisfy the same interpolation conditions. From this, with $\hat{g} \neq 0$ such that

$$\|(1 - P')\, \hat{g} \|_\infty = \| 1 - P' \| \, \| \hat{g} \|_\infty ,$$

the assumption that, for some const and some α,

$$\|(1 - P')g \|_\infty \leqslant \text{const}\, h^\alpha \, \| g^{(2)} \|_\infty \qquad \text{for all} \quad g \in C^{(2)}$$

leads to

$$\begin{aligned}\| 1 - P' \| \, \| \hat{g} \|_\infty &\leqslant \|(1 - P')\, Q'\hat{g} \|_\infty + \|(1 - Q')\, \hat{g} \|_\infty \\ &\leqslant \text{const}\, h^\alpha \, \|(Q'\hat{g})^{(2)} \|_\infty + \| 1 - Q' \| \, \| \hat{g} \|_\infty \\ &\leqslant \text{const}\, h^{\alpha-2} \, \| \hat{g} \|_\infty + \text{const} \, \| \hat{g} \|_\infty\end{aligned}$$

or

$$O(h^{-1}) = \| 1 - P' \| \leqslant \text{const}\, h^{\alpha-2} + \text{const}$$

showing that $\alpha \leqslant 1$.

PROPOSITION 2. *If* $\mathbf{t}$ *and* $\boldsymbol{\tau}$ *are related by* (2.11) *and* $\Delta t_i = h$, $i = 3,..., n$, *then, for some* const,

$$\| f - Pf \|_\infty \leqslant \text{const}\, h^2 \| f^{(3)} \|_\infty, \qquad \textit{all} \quad f \in C^{(3)}[0, 1],$$

and this estimate is sharp as regards the power of h appearing in it.

We note in passing that Daniel's main example can be treated in exactly the same way. In this example, $\mathbf{t}$ is again uniformly spaced, i.e., $\Delta t_i = h$, $i = 3,..., n$, but the points of $\boldsymbol{\tau}$ occur, with multiplicity 2, at the midpoint of every other interval. Without loss,

$$\begin{aligned} \tau_1 = t_3, \qquad \tau_i = \tau_{i+1} = (t_{i+1} + t_{i+2})/2, \qquad i = 2, 4,..., \leqslant n + 1, \\ \tau_n = t_{n+1} \qquad \text{in case } n \text{ is even.} \end{aligned} \tag{2.16}$$

For this example, the Gramian A then becomes

$$A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{16} & \frac{7}{16} & \frac{7}{16} & \frac{1}{16} & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{16} & \frac{7}{16} & \frac{7}{16} & \frac{1}{16} \\ \vdots & & & & \ddots & \vdots \end{bmatrix}$$

and one computes that

$$A((-)^i (2i - 1))_2^n = (1, -1, \tfrac{1}{2}, -1, \tfrac{1}{2},...)^T$$

so that again $\| P' \| = \| A^{-1} \|_\infty = O(h^{-1})$. Further, piecewise cubic Hermite interpolation is a simple example of a linear projector Q with the same interpolation conditions but for which $\| Q' \|$ is bounded independently of h, hence we find again that the apparent loss in the convergence rate as deduced from Lebesgue's inequality is real even when considering $f \in C^{(3)}[0, 1]$.

Quite loosely, then, the argument for Proposition 2 supports the assertion that, in case $\|P\| \to \infty$, the apparent loss in the convergence rate as deduced from Lebesgue's inequality is real only to the extent that the growth in $\|P\|$ is due to the interplay between the range of P and the interpolation conditions, i.e., some *different* choice for the range produces a "better behaved" linear projector Q. Such choice is clearly not possible in the first example.

Finally, we mention the dilemma of quadratic spline interpolation to given data, i.e., when $\boldsymbol{\tau}$ is prescribed. The second example shows that we may lose orders of convergence if we place the knots of the interpolating quadratic spline at the data points (as is usually done in odd-degree spline interpolation without such effect). On the other hand, placing the knots halfway between data points, while giving the correct order of convergence for smooth functions, shares with, say, cubic spline interpolation the disadvantage that it cannot be bounded on $C[0, 1]$ independently of $\boldsymbol{\tau}$. Marsden [5] recently showed that one could bound P on $C[0, 1]$ by 2 independently of $\boldsymbol{\tau}$ provided the knots could be so placed that the data points are halfway between knots. Unfortunately, $\mathbf{t}$ cannot be so chosen for every $\boldsymbol{\tau}$, so that Marsden's nice result is restricted to situations where it is possible to choose $\boldsymbol{\tau}$ for given $\mathbf{t}$, e.g., in the use of projectors when solving differential equations numerically (cf. Kammerer, Reddien, and Varga [4]).

3. An Exceptional Convergence Result

Recall that Daniel [3] asserted that, for quadratic spline interpolation at knots and for equispaced $\mathbf{t}$,

$$\|f - Pf\|_\infty = O(h^3) \tag{3.1}$$

for all sufficiently smooth f while Proposition 2 above stated that

$$\|f - Pf\|_\infty \leqslant \text{const}\, h^2 \|f^{(3)}\|_\infty \tag{3.2}$$

as a best possible result if f is merely known to be in $C^{(3)}[0, 1]$.

I am indebted to Blair Swartz [8] for pointing out to me that these two statements are not contradictory. The following proposition provides further evidence.

Proposition 3. *If P is quadratic spline interpolation at knots with equispaced* $\mathbf{t}$, *i.e.,* $\mathbf{t}$ *and* $\boldsymbol{\tau}$ *are related by* (2.11) *with* $\Delta t_i = h$, $i = 3,\ldots, n$, *then, for some* const,

$$\|f - Pf\|_\infty \leqslant \text{const}\, h^3(\|f^{(3)}\|_\infty + \mathrm{Var}(f^{(3)}))$$

for all $f \in \mathbb{L}_\infty^{(3)}[0, 1]$ *with* $f^{(3)}$ *of bounded variation.*

For $f \in \mathbb{L}_\infty^{(3)}[0, 1]$ with $f^{(3)}$ of bounded variation, we have by Peano's kernel theorem that

$$(f - Pf)(t) = f^{(3)}(0)\, K(t, 0) + \int_0^1 K(t, s)\, df^{(3)}(s)$$

where

$$K(\cdot, s) := (1 - P)(\cdot - s)_+^3/3!\,.$$

Let

$$f_s(t) := (t - s)_+^3$$

for some $s \in (t_{i-1}, t_i]$. Then $(Pf_s)(t) = 0$ for $t \leqslant t_i$, hence

$$\max_{t \leqslant t_i} |(f_s - Pf_s)(t)| = (t_i - s)^3 \leqslant h^3.$$

Further, for $j = i,..., n$,

$$p_j := (f_s - Pf_s)|_{[t_j, t_{j+1}]}$$

is the unique monic cubic polynomial vanishing at t_j and t_{j+1} and with

$$p_j^{(1)}(t_j) = \begin{cases} 3(t_i - s)^2, & j = i, \\ -p_{j-1}^{(1)}(t_j), & j > i. \end{cases}$$

Hence, using the fact that $\Delta t_j = h$, all j,

$$p_j(t_j + t) = -p_{j-1}(t_j - t), \qquad \text{for} \quad j > i$$

while

$$p_i(t) = (t - t_i)(t - t_{i+1})(t - t_i - a) \quad \text{with} \quad 0 \leqslant a = 3(t_i - s)^2/h \leqslant 3h.$$

Therefore,

$$\max_{t_i \leqslant t} |(f_s - Pf_s)(t)| = \max_{t_i \leqslant t \leqslant t_{i+1}} |p_i(t)| \leqslant \tfrac{3}{4}h^3$$

and so

$$\|f - Pf\|_\infty \leqslant |f^{(3)}(0)|h^3/8 + \operatorname{Var}(f^{(3)})h^3/6; \tag{3.3}$$

Q.E.D.

This proposition proves Daniel's assertion (3.1) but, at the same time, indicates its exceptional character in the extent to which special features of the interpolation scheme were used in the proof. For example, Daniel's main example does not enjoy such improvement over (3.2), a fact Daniel proves in [3, Theorem 2.3].

To prove it directly, consider interpolation to $f(t) = t^3/3!$. With $f_i' := f^{(1)}(t_i)$, all i, one verifies directly that

$$A(f_i') = (\lambda_i f^{(1)}) + (h^2/24)\begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \\ \vdots \end{pmatrix}.$$

Hence, with $e_i' := (f^{(1)} - P'f^{(1)})(t_i)$, all i, we get

$$A(e_i') = (h^2/48)[((-)^i) + 5(1^i)]$$

and therefore

$$\begin{aligned} \|(e_i')\|_\infty &= (h^2/48)\, \| A^{-1}((-)^i) + 5A^{-1}(1^i)\| \\ &\geqslant (h^2/48)(\| A^{-1} \|_\infty - 5), \end{aligned}$$

since A^{-1} takes on its norm on $((-)^i)$ while $A(1^i) = (1^i)$. It follows that for $f(t) = t^3$, $\| f^{(1)} - P'f^{(1)} \|_\infty$ is of order h and no better, hence $\| f - Pf \|_\infty$ is of order h^2 and no better even though f is analytic.

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